The Complex Propagation Constant γ

Recall that the activity along a transmission line can be expressed in terms of two functions, functions that we have described as **wave** functions:

$$V^+(z) = V_0^+ \, e^{-\gamma z}$$

$$V^-(z) = V_0^- \, e^{+\gamma z}$$

where γ is a **complex constant** that describe the properties of a transmission line. Since γ is complex, we can consider both its **real** and **imaginary** components.

$$\boxed{\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \doteq \alpha + j\beta}$$

where $\alpha = \text{Re}\{\gamma\}$ and $\beta = \text{Im}\{\gamma\}$. Therefore, we can write:

$$V^+(z) = V_0^+ \, e^{-\gamma z} = V_0^+ \, e^{-(\alpha + j\beta)z} = V_0^+ \, e^{-\alpha z} \, e^{-j\beta z}$$

Q: *What **are** these constants α and β? What do they **physically** represent?*

A: Remember, a complex value can be expressed in terms of its **magnitude** and **phase**. For example:

$$V_0^+ = \left|V_0^+\right| e^{j\phi_0^+}$$

Likewise:

$$V^+(z) = \left|V^+(z)\right| \; e^{j\,\phi^+(z)}$$

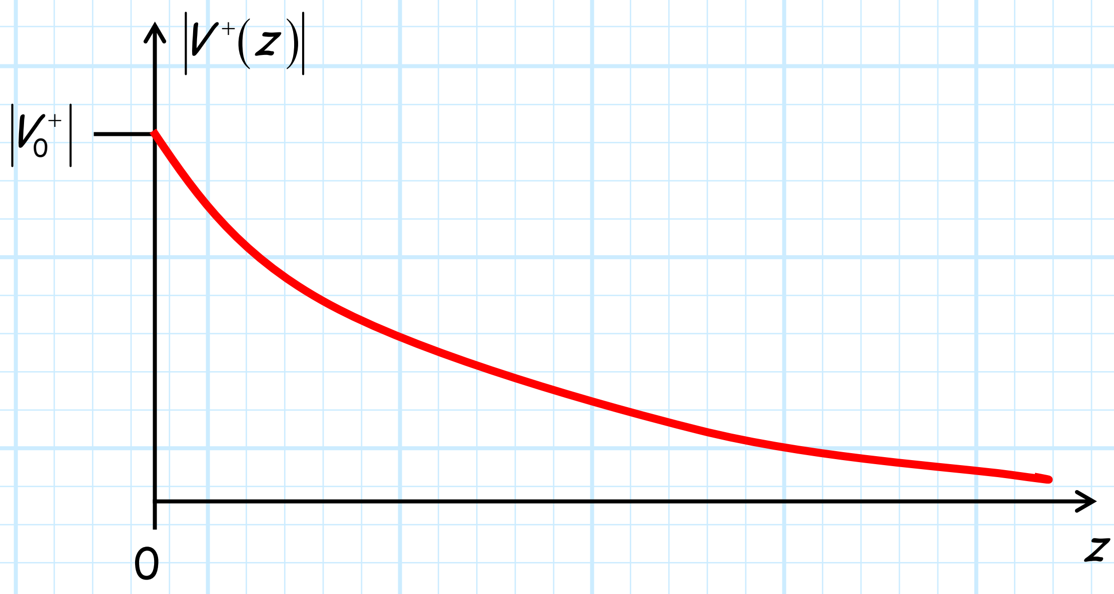
And since:

$$\begin{aligned} V^+(z) &= V_0^+ \; e^{-\alpha z} \; e^{-j\beta z} \\ &= \left|V_0^+\right| e^{j\phi_0^+} \; e^{-\alpha z} \; e^{-j\beta z} \\ &= \left|V_0^+\right| \, e^{-\alpha z} \; e^{j\left(\phi_0^+ - \beta z\right)} \end{aligned}$$

we find:

$$\left|V^+(z)\right| = \left|V_0^+\right| e^{-\alpha z} \qquad\qquad \phi^+(z) = \phi_0^+ - \beta z$$

It is evident that $e^{-\alpha z}$ **alone** determines the **magnitude** of wave $V^+(z) = V_0^+ \, e^{-\gamma z}$ as a function of position z.



Therefore, α expresses the **attenuation** of the signal due to the loss in the transmission line. The larger the value of α, the greater the exponential attenuation.

Q: So *what **is** the constant β? What does **it** physically mean?*

A: Recall

$$\phi^+(z) = \phi_0^+ - \beta z$$

represents the relative **phase** of wave $V^+(z)$; a **function** of transmission line **position** z. Since phase ϕ is expressed in **radians**, and z is distance (in meters), the value β must have **units** of:

$$\beta = \frac{\phi}{z} \qquad \frac{\text{radians}}{\text{meter}}$$

Thus, if the value β is **small**, we will need to move a **significant distance** Δz down the transmission line in order to observe a change in the relative phase of the oscillation.

Conversely, if the value β is **large**, a significant change in relative phase can be observed if traveling a **short** distance $\Delta z_{2\pi}$ down the transmission line.

Q: *How far must we move along a transmission line in order to observe a change in relative phase of 2π radians?*

A: We can easily determine this distance ($\Delta z_{2\pi}$, say) from the transmission line characteristic β.

$$2\pi = \phi(z + \Delta z_{2\pi}) - \phi(z) = \beta\,\Delta z_{2\pi}$$

or, rearranging:

$$\Delta z_{2\pi} = \frac{2\pi}{\beta} \qquad \Rightarrow \qquad \beta = \frac{2\pi}{\Delta z_{2\pi}}$$

The **distance** $\Delta z_{2\pi}$ over which the relative phase changes by 2π **radians**, is more specifically known as the **wavelength** λ of the propagating wave (i.e., $\lambda \doteq \Delta z_{2\pi}$):

$$\boxed{\lambda = \frac{2\pi}{\beta} \qquad \Rightarrow \qquad \beta = \frac{2\pi}{\lambda}}$$

The value β is thus essentially a **spatial frequency**, in the same way that ω is a **temporal** frequency:

$$\omega = \frac{2\pi}{T}$$

Note T is the **time** required for the phase of the oscillating signal to change by a value of 2π radians, i.e.:

$$\omega T = 2\pi$$

And the **period** of a sinewave, and related to its **frequency** in Hertz (cycles/second) as:

$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

Compare these results to:

$$\beta = \frac{2\pi}{\lambda} \qquad 2\pi = \beta\lambda \qquad \lambda = \frac{2\pi}{\beta}$$

Q: *So, just how **fast** does this wave propagate down a transmission line?*

We describe wave velocity in terms of its **phase velocity**—in other words, how **fast** does a specific value of absolute phase ϕ seem to **propagate** down the transmission line.

Since velocity is change in distance with respect to **time**, we need to first express our propagating wave in its real form:

$$\begin{aligned} v^+(z,t) &= Re\left\{V^+(z)\,e^{-j\omega t}\right\} \\ &= \left|V_0^+\right| \cos\left(\omega t - \beta z + \phi_0^+\right) \end{aligned}$$

Thus, the absolute phase is a function of **both** time and frequency:

$$\phi^+(z,t) = \omega t - \beta z + \phi_0^+$$

Now let's set this phase to some **arbitrary** value of ϕ_c radians.

$$\omega t - \beta z + \phi_0^+ = \phi_c$$

For **every** time t, there is **some** location z on a transmission line that has this phase value ϕ_c. That location is evidently:

$$z = \frac{\omega t + \phi_0^+ - \phi_c}{\beta}$$

Note as **time increases**, so to does the **location** z on the line where $\phi^+(z,t) = \phi_c$.

The **velocity** v_p at which this phase point moves down the line can be determined as:

$$v_p = \frac{dz}{dt} = \frac{d\left(\dfrac{\omega t + \phi_0^+ - \phi_c}{\beta}\right)}{dt} = \frac{\omega}{\beta}$$

This wave velocity is the **velocity of the propagating wave**!

Note that the value:

$$\frac{v_p}{\lambda} = \frac{\omega}{\beta}\frac{\beta}{2\pi} = \frac{\omega}{2\pi} = f$$

and thus we can conclude that:

$$v_p = f\lambda$$

as well as:

$$\beta = \frac{\omega}{v_p}$$

Q: *But these results were derived for the $V^+(z)$ wave; what about the* ***other*** *wave $V^-(z)$?*

A: The results are essentially the **same**, as each wave depends on the same value β.

The only **subtle difference** comes when we evaluate the phase velocity. For the wave $V^{-}(z)$, we find:

$$\phi^{-}(z,t) = \omega t + \beta z + \phi_0^{-}$$

Note the **plus sign** associated with βz!

We thus find that some arbitrary phase value will be located at location:

$$z = \frac{-\phi_0^{-} + \phi_c - \omega t}{\beta}$$

Note now that an **increasing time** will result in a **decreasing** value of **position** z. In other words this wave is propagating in the direction of decreasing position z—in the **opposite** direction of the $V^{+}(z)$ wave!

This is **further** verified by the derivative:

$$v_p = \frac{dz}{dt} = \frac{d\left(\dfrac{-\phi_0^{-} + \phi_c - \omega t}{\beta}\right)}{dt} = -\frac{\omega}{\beta}$$

Where the **minus sign** merely means that the wave propagates in the $-z$ direction. Otherwise, the **wavelength** and **velocity** of the two waves are **precisely** the same!